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COMPARISON OF THE EFFECTIVENESS OF MINIMIZING COST FUNCTION PARAMETERS FOR ACTIVE CONTROL OF VIBRATIONAL ENERGY TRANSMISSION IN A LIGHTLY DAMPED STRUCTURE

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The success of an active control of vibration system depends upon both the cost function used and the positions of the controlling actuators. The cost function used also affects the best actuator positions since their performance is judged on the attenuation of this parameter. However, the physical success will be dependent on how well the cost function represents the actual physical vibration. Sometimes the most meaningful cost function can be calculated in a theoretical model but is difficult to measure in practice, and a compromise to a more practical one is often made. In this paper four cost functions are considered with the aim of reducing the vibration transmitted from the base to the end of a lightweight cantilever two-dimensional structure, and their performances compared with a view to evaluating the true success in using other cost function parameters in reducing the vibrational energy.

Of the four cost functions studied, two are energy-based: one representing the total vibrational energy and one using only the flexural energy level. The other two cost functions are based on velocity measurements: the sum of the squares of the translational velocity components, and one additionally using rotational velocity measurements. An initial study confirms that the total vibrational energy is the cost function which most comprehensively represents the beam vibration and is used as the reference in a comparison of the other cost functions.

Then, a ranking of the best actuator positions on the structure is determined to achieve the best reductions in each cost function. For each of these sets of actuator positions the consequential attenuation in the total vibrational energy is evaluated whilst minimizing the other cost functions. Thus, the effectiveness of these cost functions in reducing the total vibrational energy is evaluated.

1. INTRODUCTION

Unwanted vibrations can have many undesirable effects in structures. They can cause damage to the structure or an adjoining component. To a lesser extent they may prevent the structure being used for its intended purpose if too much vibrational energy exists in critical regions. For the lightweight structure studied here, a typical application might be to mount an antenna on a satellite using a boom-arm, and the focus of the optimization is to reduce the base vibration being transmitted to the antenna at the end of the arm. Traditional techniques of reducing vibration, by adding additional mass or damping, are not always suitable because the issue of weight is often important.

An alternative strategy is the addition of a control mechanism to an existing structure. One such technique is that of active vibration control, which uses secondary sources of

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vibration in order to reduce the original vibration by destructive interference at some desired region. This is now a well-proven technique [1]. The vibration, which is represented by some suitable parameter, is minimized by the control system. Sometimes the parameter is compromised on practical grounds. Originally, this parameter simply represented the magnitude of the vibration in the region of one or a number of strategic points [1]; however, the use of a representation of power flow was soon seen as a more effective practice. In general, structures are lightly damped and therefore the mechanical impedance is strongly dependent upon the frequency and also upon the positions of the sensors and actuators on a structure. Therefore, a single measurement of velocity or force is not a sufficient representation of power. Earlier use of power [2] demonstrates the advantage of using a power measurement, despite the added complexity of such systems. Howard and Hansen [3] have shown that if either one of the force or the acceleration are minimized as a cost function for active vibration isolation, this does not necessarily lead to the minimization of the other. Pan and Hansen [4] demonstrate that the use of acceleration as a measurement to reduce power flow along a beam is sufficient if the sensor is placed outside the near field of any power sources. Power is used as the cost function parameter for vibration isolation by Bardou *et al.* [5], who compare different types of strategy used (to minimize power supplied by the primary source or maximize power absorbed by the secondary sources). Brennan *et al.* [6] show that the best power measurement strategy can depend upon the nature of the problem. The application of feedforward active control is used here, its application to the structure considered here was discussed in reference [7], where the advantage of using an energy-based cost function over one based on velocities at a point was also given.

The position of the actuators on structures is also important if the maximum obtainable reduction is sought, and this is often a discrete optimization problem. Optimal actuator positions can be found by exhaustive search if combinatorially feasible, but other techniques such as evolutionary algorithms have been successfully employed; for example see reference [8] or reference [9]. In this paper the number of possible actuator combinations is relatively small and an exhaustive search is feasible. The method of determination of the optimal actuator positions is not given here, but the authors have previously detailed the exhaustive search method used to find the best actuator positions for one of the cost functions considered here [7].

This paper is organized as follows: Section 2 details the two-dimensional (2-D) structure considered and presents an overview of the application of active vibration control systems to the structure. Section 3 introduces the parametric measurements required in the application of active control, and in section 4 the four different cost functions using these measurements are derived. A single-frequency comparison between cost functions is reported in section 5, which provides an introduction to the differing levels of success attainable using different cost functions. The total vibrational energy is then used as a reference to compare the average performance of all other three cost functions, over a band of frequencies. This is reported in section 6.

2. THE STRUCTURE

The structure studied is shown in Figure 1, the co-ordinate units are in metres. The structure is the same used previously by the authors [7, 10]: a lightweight cantilever structure comprising 40 rigidly joined beams of lengths 1 and 1·414 m. The individual beam parameters correspond to aluminium beams of approximate rectangular cross-sectional dimensions 50 mm × 25 mm, with the longer dimension in the x–y plane. The structure is

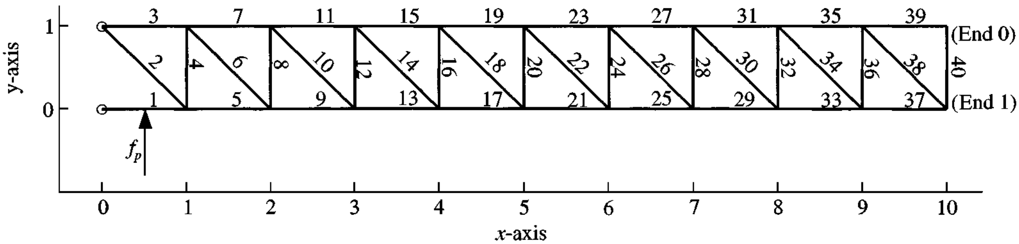

Figure 1. Diagram of the structure considered showing; global co-ordinates, primary force input and beam numbering.

two-dimensional; motion is only considered in the x–y plane. The rightmost beam is the subject of the vibration minimization, and is referred to as beam 40, as labelled in Figure 1.

The analytical model, which is described by the authors in reference [10] and in more detail in reference [11], considers both axial and flexural beam vibration. All the cost functions considered are derived from the inter-beam coupling force and the velocity components at the ends of the beam. In the application of active vibration control, secondary control forces are applied to points on the structure in order to reduce the effect of the primary force, and produce a net reduction in the vibration in the end beam. The secondary forces used are applied by means of double-acting actuators which are placed in or alongside a beam and produce equal and opposite axial forces at each end.

In practice, the active control system adaptively seeks the complex secondary force control signals which cause the vibration in the end beam to be minimized for each frequency. The control aim is to find the minimum of a multi-dimensional quadratic surface. The optimal value of the control forces can be found analytically and hence the minimum cost function value can be predicted theoretically. This is dependent upon the mechanical coupling between both the primary force, the secondary forces and the locations where the cost function parameter is measured.

3. APPLICATION OF ACTIVE VIBRATION CONTROL TO STRUCTURE

The base vibration is modelled as a single sinusoidal transverse force 1 N applied at the middle of one of the beams adjoined to the base (as shown in Figure 1). In active control terminology this is called the *primary force*. Two vectors defining the complex force and velocity components (in all the degrees of freedom considered) at the joints at the ends of beam 40 in the absence of any other forces (i.e., without active control operative), are denoted $\mathbf{f}_p$ and $\mathbf{v}_p$. Active control applies *secondary forces* to "counter" vibrations on the structure. Their effect is determined by a vector describing the complex values of secondary forces of each actuator $\mathbf{f}_s$, and either a "transformed" force or mobility transfer matrix ($\mathbf{C}$ or $\mathbf{Y}$). This represents the resultant force or velocity components from the secondary forces at the joints at the ends of beam 40, in all three degrees of freedom resulting from the axial forces of the actuators. More detailed information on the form of $\mathbf{C}$ and $\mathbf{Y}$ is given in reference [7]. The net force and velocity vectors from the combination of both primary and secondary forces is then given by the sum of these two components, so that

$$\mathbf{f} = \mathbf{f}_p + \mathbf{C}\mathbf{f}_s, \quad (1)$$

where the format of the force vector $\mathbf{f}$ is given by

$$\mathbf{f} = [\{f_x^{40,0}\ f_y^{40,0}\ f_\theta^{40,0}\}\ \{f_x^{40,1}\ f_y^{40,1}\ f_\theta^{40,1}\}]^{\mathrm{T}}. \tag{2}$$

Other force and velocity vectors used ($\mathbf{f}$, $\mathbf{f}_p$, $\mathbf{v}$, $\mathbf{v}_p$) all have the some format, with elements which define the translational forces or velocities in the x and y directions, and the moments or rotational velocities, at end 0 and end 1 of beam 40. $\mathbf{f}_s$ is a vector comprising individual complex secondary actuator forces. The net velocity vector $\mathbf{v}$ is similarly,

$$\mathbf{v} = \mathbf{v}_p + \mathbf{Y}\mathbf{f}_s, \tag{3}$$

where $\mathbf{v}_p$ is the vector of the six velocity components due to the primary force only (the velocity vectors being of the same format as the force vectors). At each end of beam 40 the force and velocity are totally described in the 2-D model by two translational components and one rotary component; x, y and θ, in the co-ordinate system indicated in Figure 1.

4. COST FUNCTION PARAMETERS

The parameter used as the objective function (or *cost function* in active control terminology) in minimized by the control system. The physical success of the control systems depends, in part, on how well the cost function represents the unwanted physical vibration. For example, a measurement of acceleration or velocity at a single point only represents the vibration at that single point, and may not be a good measure of energy flow into a beam. It may not always be practical to obtain a direct measure of the desired parameter to be controlled, however (for example, dissipated power in a beam), though this may easily be calculated from a theoretical model. Four cost functions are studied here, all with the aim of reducing some measure of vibration in the end beam of the structure. There are two types of functions: energy-based cost functions utilizing both force and velocity measurements, and velocity-based cost functions. The former generally gives a good measure of performance but is more difficult to measure in practice, as inter-beam coupling force measurements are required.

4.1. MINIMIZATION OF BEAM FLEXURAL ENERGY

This cost function is equal to the energy level in beam 40 due to its flexural vibration. This parameter has previously been used by Keane [12] and the authors [10] to reduce the vibrational energy on the structure considered here, using genetic algorithm optimization of the geometry, and also for the optimum placement of actuators [7]. The flexural energy level in the beam arises as a result of the balance between the average energy flowing into the beam at its ends, and the average dissipation of energy due to its damping.

For harmonic vibration, the average dissipated power is defined as half of the real part of the conjugate product of the complex force and velocity vectors at the joints at the ends of the beam,

$$J_P = \tfrac{1}{2}\,\mathrm{Re}\{\mathbf{f}^{\mathrm{H}}\mathbf{v}\}, \tag{4}$$

which can be more conveniently expressed in the form

$$J_P = \tfrac{1}{4}\,(\mathbf{f}^{\mathrm{H}}\mathbf{v} + \mathbf{v}^{\mathrm{H}}\mathbf{f}). \tag{5}$$

Using equations (1) and (3) this can be expressed in terms of $\mathbf{f}_s$, the independent variable for the cost function minimization,

$$J_P = \tfrac{1}{4}\,[\mathbf{f}_s^{\mathrm{H}}(\mathbf{C}^{\mathrm{H}}\mathbf{Y} + \mathbf{Y}^{\mathrm{H}}\mathbf{C})\,\mathbf{f}_s + \mathbf{f}_s^{\mathrm{H}}(\mathbf{C}^{\mathrm{H}}\mathbf{v}_p + \mathbf{Y}^{\mathrm{H}}\mathbf{f}_p) + (\mathbf{f}_p\mathbf{Y} + \mathbf{v}_p\mathbf{C})\,\mathbf{f}_s + \mathbf{f}_p^{\mathrm{H}}\mathbf{v}_p + \mathbf{v}_p^{\mathrm{H}}\mathbf{f}_p], \quad (6)$$

which can be written in a general quadratic form,

$$J = \mathbf{x}^{\mathrm{H}}\mathbf{A}\mathbf{x} + \mathbf{x}^{\mathrm{H}}\mathbf{b} + \mathbf{b}^{\mathrm{H}}\mathbf{x} + c. \quad (7)$$

The positive scalar c represents the value of the cost function due to the primary excitation only (without active control; $\mathbf{x} = \mathbf{0}$). The $\mathbf{x}^{\mathrm{H}}\mathbf{A}\mathbf{x}$ term represents the value of the cost function due to the secondary source excitation only (without a primary source of structural excitation), and this is obviously always positive (unless there is an external power input into beam 40). Based on these physical grounds, $\mathbf{A}$ will always be positive definite (see Appendix A). This is verified in practice by confirming that all the eigenvalues of $\mathbf{A}$ are positive. Thus, the derivation of the minimum value of the cost function can be greatly simplified. Also, as the active control system is *over-determined* (there are more degrees of freedom for sensors than actuators), $\mathbf{A}$ is of full rank. The minimization of the quadratic form in equation (7) is detailed in Appendix A. This yields the optimum secondary control vector

$$\mathbf{x}_o = -\,\mathbf{A}^{-1}\mathbf{b} \quad (8)$$

and, therefore, the optimum secondary force vector is

$$\mathbf{f}_{s_o} = -\,(\mathbf{C}^{\mathrm{H}}\mathbf{Y} + \mathbf{Y}^{\mathrm{H}}\mathbf{C})^{-1}(\mathbf{C}^{\mathrm{H}}\mathbf{v}_p + \mathbf{Y}^{\mathrm{H}}\mathbf{f}_p). \quad (9)$$

From Appendix A the minimized value of the dissipated power is of the form

$$J_o = c - \mathbf{b}^{\mathrm{H}}\mathbf{A}^{-1}\mathbf{b}. \quad (10)$$

Hence the minimized net dissipated power is explicitly,

$$J_{P_o} = \tfrac{1}{4}\,[(\mathbf{f}_p^{\mathrm{H}}\mathbf{v}_p + \mathbf{v}_p^{\mathrm{H}}\mathbf{f}_p) - (\mathbf{f}_p^{\mathrm{H}}\mathbf{Y} + \mathbf{v}_p^{\mathrm{H}}\mathbf{C})\,(\mathbf{C}^{\mathrm{H}}\mathbf{Y} + \mathbf{Y}^{\mathrm{H}}\mathbf{C})^{-1}(\mathbf{C}^{\mathrm{H}}\mathbf{v}_p + \mathbf{Y}^{\mathrm{H}}\mathbf{f}_p)]. \quad (11)$$

The average power dissipated in beam 40 is simply related to the average energy level of the beam, thus,

$$E_{flex} = \frac{J_P}{c_d}, \quad (12)$$

where c_d is the beam damping which has the value $20\ \mathrm{s}^{-1}$ at all frequencies, in this case. Even though minimizing either E_{flex} or J_P will result in the same optimum secondary force vector, the flexural energy is used here so that it can subsequently be summed with the rigid-body kinetic energy below.

4.2. THE RIGID-BODY KINETIC ENERGY OF A BEAM

The minimization of the flexural energy in the beam, calculated above, only accounts for the motion of the beam due to its flexure. If the beam does not undergo flexure, its power dissipation and therefore the flexural energy is zero. However, the beam may still move as a rigid body and this motion would not be detected by E_{flex}. So, even though E_{flex} may have been reduced to its minimum value, a significant amount of undetected rigid-body motion may exist, which could dominate the motion of the beam, or any object connected to it. Therefore, a cost function was sought which represents all the beam energy due to its motion; the flexural energy level and the *rigid-body kinetic energy*, E_{rigid}. The minimization of this *total vibrational energy* cost function would therefore be superior and achieve the best overall vibration reduction.

Considering the rigid-body kinetic energy due to movement in the axial direction of the beam 40 (in the global co-ordinates of Figure 1, in the y-axis direction) the velocity of the centre of mass of the beam, v_{cm}, is given by the average of the y-axis velocities at the end of the beam. At beam ends 0 and 1 the velocities are

$$v^0_{cm_y} = \mathrm{Re}\{V^0_y\,\mathrm{e}^{\mathrm{i}\omega t}\}, \quad v^1_{cm_y} = \mathrm{Re}\{V^1_y\,\mathrm{e}^{\mathrm{i}\omega t}\}, \tag{13a,b}$$

where V^0_y and V^1_y are the complex amplitudes. The instantaneous rigid-body kinetic energy is thus described by

$$KE_y(t) = \frac{1}{2}mv^2_{cm}(t) = \frac{1}{2}m\left(\frac{v^0_{cm_y}(t) + v^1_{cm_y}(t)}{2}\right)^2, \tag{14}$$

where m is the total mass of the beam. For harmonic excitation the total time-averaged kinetic energy is given by

$$\overline{KE_y} = \frac{m}{16}\,\mathrm{Re}\{V^0_yV^{0*}_y + 2V^0_yV^{1*}_y + V^1_yV^{1*}_y\}. \tag{15}$$

The rigid-body kinetic energy due to the translation of the centre of mass of the beam in its transverse sense, in the x-axis direction, can be expressed in terms of the scaled real part of the product of the x-axis velocities,

$$\overline{KE_{trans_x}} = \frac{m}{16}\,\mathrm{Re}\{V^0_xV^{0*}_x + 2V^0_xV^{1*}_x + V^1_xV^{1*}_x\}. \tag{16}$$

When the beam rotates as a rigid body about its centre of mass the rotational kinetic energy is

$$KE_{rot} = \tfrac{1}{2}\,I\dot{\theta}^2, \tag{17}$$

where I is the second moment of mass of the beam about its centre and $\dot{\theta}$ is the angular velocity of the beam. For small θ, the instantaneous kinetic energy can be expressed in terms of end velocities, $v^0_{cm_x}$ and $v^1_{cm_x}$,

$$KE_{rot_x}(t) = \frac{mL^2}{24}\left(\frac{v^0_{cm_x}(t) - v^1_{cm_x}(t)}{L}\right)^2, \tag{18}$$

where L is the beam length. For harmonic excitation the average kinetic energy can be expressed by

$$\overline{KE_{rot_x}} = \frac{m}{48}\,\mathrm{Re}\{V^0_xV^{0*}_x - 2V^0_xV^{1*}_x + V^1_xV^{1*}_x\}. \tag{19}$$

Hence, the total rigid-body energy of the beam, E_{rigid}, is obtained from the sum of equations (15), (16) and (19), so

$$E_{rigid} = \frac{m}{48}\,\mathrm{Re}\{3V_y^0V_y^{0*} + 6V_y^0V_y^{1*} + 3V_y^1V_y^{1*} + 4V_x^0V_x^{0*} + 4V_x^0V_x^{1*} + 4V_x^1V_x^{1*}\}. \quad (20)$$

After some manipulation, E_{rigid} can be expressed succinctly in a linear matrix formulation using the velocity component scaling matrix **N**,

$$E_{rigid} = \frac{m}{96}\,\mathbf{v}^{\mathrm{H}}\mathbf{N}\mathbf{v}, \quad (21)$$

where **v** is the velocity vector (3), and

$$\mathbf{N} = \begin{pmatrix} 8 & 0 & 0 & 4 & 0 & 0 \\ 0 & 6 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 8 & 0 & 0 \\ 0 & 6 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (22)$$

Expanding (21) using (3) the cost function can be expressed in quadratic form,

$$E_{rigid} = \frac{m}{96}\,(\mathbf{f}_s\mathbf{Y}^{\mathrm{H}}\mathbf{N}\mathbf{Y}\mathbf{f}_s + \mathbf{f}_s^{\mathrm{H}}\mathbf{Y}^{\mathrm{H}}\mathbf{N}\mathbf{v}_p + \mathbf{v}_p^{\mathrm{H}}\mathbf{N}\mathbf{Y}\mathbf{f}_s + \mathbf{v}_p^{\mathrm{H}}\mathbf{N}\mathbf{v}_p). \quad (23)$$

4.3. MINIMIZING THE TOTAL VIBRATIONAL ENERGY OF THE BEAM

A global cost function is defined which is the *total vibrational energy* of beam 40, E_{total}, combining flexural energy and rigid-body kinetic energy,

$$E_{total} = E_{flex} + E_{rigid}. \quad (24)$$

This is the sum of the two quadratic functions (6) (using equation (12) to convert to energy) and equation (23), resulting in another quadratic form, which define the coefficients in equation (7) as

$$\mathbf{A} = \frac{1}{4c_d}(\mathbf{C}^{\mathrm{H}}\mathbf{Y} + \mathbf{Y}^{\mathrm{H}}\mathbf{C}) + \frac{m}{96}\,\mathbf{Y}^{\mathrm{H}}\mathbf{N}\mathbf{Y}, \quad (25a)$$

$$\mathbf{b} = \frac{1}{4c_d}(\mathbf{C}^{\mathrm{H}}\mathbf{v}_p + \mathbf{Y}^{\mathrm{H}}\mathbf{f}_p) + \frac{m}{96}\,\mathbf{Y}^{\mathrm{H}}\mathbf{N}\mathbf{v}_p, \quad (25b)$$

$$c = \frac{1}{4c_d}(\mathbf{f}_p^{\mathrm{H}}\mathbf{v}_p + \mathbf{v}_p^{\mathrm{H}}\mathbf{f}_p) + \frac{m}{96}\,\mathbf{v}_p^{\mathrm{H}}\mathbf{N}\mathbf{v}_p. \quad (25c)$$

The optimum secondary force vector and the minimized value of E_{total} are given by equations (8) and (10) with the values of **A**, **b** and c as defined in equation (25).

The minimum value of the cost function is obtained if **A** is positive definite. The first term is quadratic as **N** is real symmetric and hence positive definite. The second term is semipositive definite for all secondary actuator positions except on beam 40 (see section 4.1). The sum of these two terms results in a positive-definite function.

4.4. MINIMIZATION OF THE SUM OF THE SQUARES OF THE TRANSLATIONAL JOINT VELOCITIES

The first velocity-based cost function studied, J_{trans}, uses the sum of the squares of the translational velocity components at the ends of beam 40. These measurements can be readily obtained using standard accelerometers with the relevant orientations. To be consistent with the cost function derived in the following section, this cost function is scaled so that it is equal to the sum of the rigid-body kinetic energies of each half-beam length of beam 40. The time-averaged values of kinetic energy at end 0, for example, of beam 40 in the x- and y-axis directions are, therefore,

$$\overline{KE_x^0} = \frac{m}{8}\,|V_x^0|^2, \quad \overline{KE_y^0} = \frac{m}{8}\,|V_y^0|^2, \tag{26a,b}$$

for harmonic excitation, and adhering to the previous notation used in equation (13). A reduced velocity vector, containing only translational components, may be achieved by premultiplying the velocity defined in equation (3) with the matrix **P**,

$$\mathbf{P} = \frac{m}{8}\,\mathrm{diag}(1 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; 0). \tag{27}$$

The cost function J_{trans} is then

$$J_{trans} = \mathbf{v}^{\mathrm{H}}\mathbf{P}\mathbf{v}. \tag{28}$$

Expanding equation (28) with equation (3) results in a quadratic function of the form (7) where

$$\mathbf{A} = \mathbf{Y}^{\mathrm{H}}\mathbf{P}\mathbf{Y}, \tag{29a}$$

$$\mathbf{b} = \mathbf{Y}^{\mathrm{H}}\mathbf{P}\mathbf{v}_p, \tag{29b}$$

$$c = \mathbf{v}_p^{\mathrm{H}}\mathbf{P}\mathbf{v}_p. \tag{29c}$$

The minimum cost function, and the optimum secondary force vector is given in equations (8) and (10) with the values of **A**, **b** and c as defined in equation (29).

4.5. MINIMIZATION OF THE WEIGHTED SUM OF THE SQUARES OF ALL VELOCITY COMPONENTS

In order to provide a more comprehensive velocity-based cost function, the angular velocity at each joint could also be measured. Even though devices to measure angular velocity are not as commonplace as their translational counterparts, low-cost practical devices are readily available.

Intuitively, it is a good strategy to reduce all the velocity components at the ends of the beam (ideally to zero). A cost function that pursues this aim is the sum of the squares of all velocity components. However, the arbitrary combination of the squares of the translational and rotary components will produce a cost function in which the relative "weighting" between these two different quantities will depend on the system of units (e.g., CGS, SI, etc.) in which the cost function is defined. Whilst it is not possible to define this weighting rigorously for anything other than solely rigid-body motion, an attempt is made to produce a sensible weighting. This weighting is achieved by considering the kinetic energy represented by both the linear and rotational velocity components. This cost function is easier to implement in practice than the total energy cost function, since the measurement of flexural energy requires the inter-beam coupling forces, which are not as easily obtained as a velocity measurement, especially more so if the application of active control was an "add-on" to an existing structure.

To determine a sensible weighting the beam is considered as two half-lengths. The halves are assumed to move as rigid-body levers whilst being hinged about the joints at the beam ends. Each translational velocity component is then assumed to represent the kinetic energy of a lumped mass equal to the mass of half of beam 40. Each rotational velocity component is assumed to represent the kinetic energy due to the rotation of the distributed mass of each half-beam length "lever". This may appear to disregard the flexural motion of the beam; however in the frequency region considered only the first transverse mode is significant. Considering the beam as two "rigid-body" halves allows the first transverse mode to be approximated, giving some credence to this approximation.

The kinetic energy of each half-length of beam 40 due to the translation in the x- and y-axis directions is as given above in the derivation of the cost function J_{trans}, (equation (26)). Considering the average rotational kinetic energy of one-half of beam 40 with distributed mass, this is represented using the rotational velocity component at the beam end. So, at end 0 this is given by

$$\overline{KE_\theta^0} = \frac{m}{96}\,|V_\theta^0|^2. \tag{30}$$

The relative scaling between the translational and rotational components is therefore shown in equations (26) and (30). A diagonal pre-multiplying matrix **L** allows all the components of the velocity squared cost function J_{all} to be written as

$$J_{all} = \mathbf{v}^{\mathrm{H}}\mathbf{L}\mathbf{v}, \tag{31}$$

where **L** is defined as

$$\mathbf{L} = \frac{m}{8}\,\mathrm{diag}\left(1 \quad 1 \quad \frac{1}{12} \quad 1 \quad 1 \quad \frac{1}{12}\right). \tag{32}$$

Expanding equation (31) with equation (3) results in a quadratic function of the form (7) where

$$\mathbf{A} = \mathbf{Y}^{\mathrm{H}}\mathbf{L}\mathbf{Y}, \tag{33a}$$

$$\mathbf{b} = \mathbf{Y}^{\mathrm{H}}\mathbf{L}\mathbf{v}_p, \tag{33b}$$

$$c = \mathbf{v}_p^{\mathrm{H}}\mathbf{L}\mathbf{v}_p. \tag{33c}$$

The minimum cost function, and the optimum secondary force vector is given in equations (8) and (10) with the values of **A**, **b** and c as defined in equation (33).

5. THE USE OF DIFFERENT COST FUNCTIONS AT SINGLE FREQUENCIES

As shown in the previous section, the use of E_{flex} alone does not give a proper representation of all the vibrational energy of the beam. The authors have previously used E_{flex} to study the reduction of the vibration transmission of the structure shown in Figure 1, using both passive and active vibration control methods [7, 10]. To investigate the potential consequences of using E_{flex} as the cost function instead of E_{total} a comparison was conducted, which was also extended to the two velocity-based cost functions developed above. To demonstrate that the choice of cost function can have important consequences on the success of an active control system, two single-frequency scenarios are presented ahead of the full analysis. The first case is that using an actuator on beam 3 of the structure at a frequency of 170 Hz. Firstly, considering the two energy-based cost functions, Figure 2 shows the effect on the total vibrational energy of the beam (E_{total}) when using E_{flex} and E_{total} as the cost functions. Both the constituent rigid and flexural energy components are also shown. It is seen that in minimizing E_{flex} an increase in the value of E_{rigid} is seen, which then becomes the dominant component of E_{total}, such that beyond a certain point reductions in E_{flex} are fruitless. However, when E_{total} is used as the cost function the minimum value of E_{total} is thus achieved, even though a small increase in E_{rigid} occurs.

The results of this comparison are summarized in Table 1, which also details the reductions in all of the other parameters considered when each is minimized (used as the cost function). The Table also includes the results for a second case (Case 2, of using two

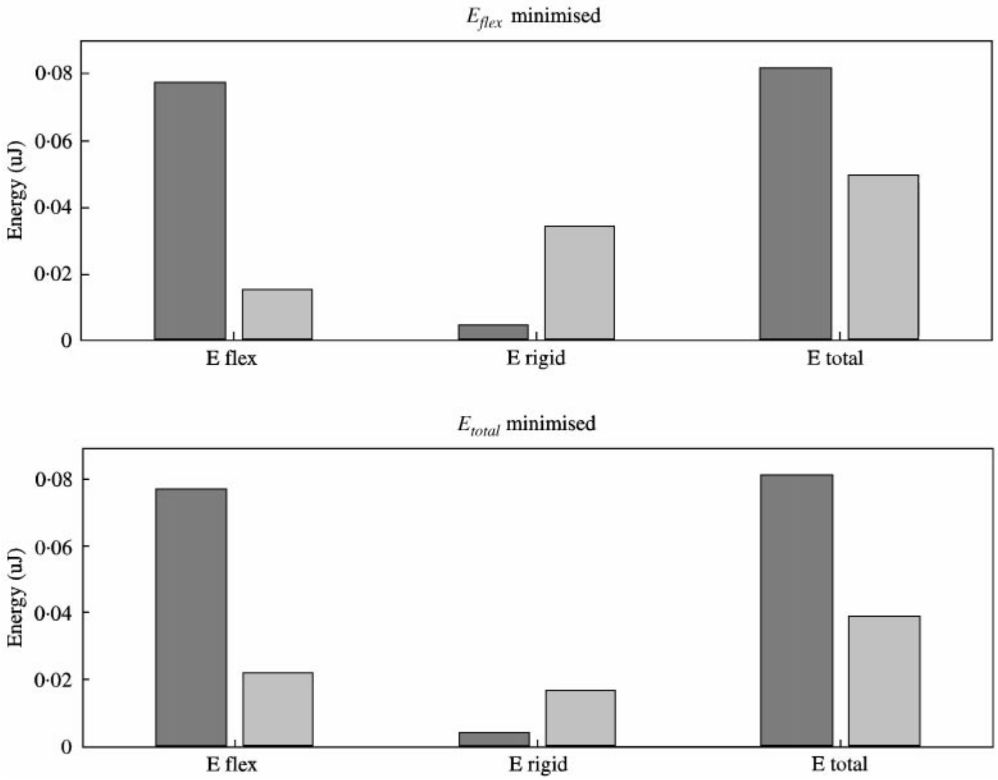

Figure 2. Effect of applying active control on E_{flex}, E_{rigid} and E_{total} of beam 40 for Case 1 with (a) E_{flex} and (b) E_{total} used as cost function (shading scheme: Dark: no active control, Light: active control applied).

TABLE 1

Summary of results showing the effect on the values of four parameters and E_{rigid}, when each parameters is minimized as an active control (AVC) cost function for two sets of actuator positions at different frequencies

Case no.	Actuator positions	Frequency (Hz)	AVC cost function	Attenuation achieved in each parameter by minimizing cost function shown (dB)				
				E_{total}	E_{flex}	E_{rigid}	J_{trans}	J_{all}
1	3	170	E_{total}	3·2	5·5	− 6·2	− 6·0	− 3·3
			E_{flex}	2·2	7·1	− 9·2	− 9·0	− 5·0
			J_{trans}	0·3	0·3	0·07	0·05	− 0·1
			J_{all}	− 2·1	− 1·9	− 4·2	− 4·0	1·3
2	5, 19	160	E_{total}	6·4	6·4	6·5	6·6	8·4
			E_{flex}	4·6	7·7	− 0·7	− 0·5	− 0·5
			J_{trans}	5·1	4·3	45·9	41·0	3·5
			J_{all}	6·2	5·9	8·0	8·3	9·2

actuators on beams 5 and 9 at a frequency of 160 Hz. It is again seen here that the use of E_{total} as the cost function is superior to E_{flex}. In this particular case the use of E_{flex} increases the rigid-body kinetic energy, whilst the use of E_{total} reduces it by about 6 dB with less than 1·5 dB being sacrificed for the reduction in the value of E_{flex}. Also, for Case 2, using either J_{trans} or J_{all} as the cost function yields good reductions in E_{total}, which are better than those obtained using E_{flex} as the cost function. Here J_{trans} is seen to achieve substantial reductions in the rigid-body kinetic energy, but it is the smaller reduction in E_{flex} in this case which makes the reduction achieved in E_{total} second to that for J_{all}. The result of using J_{trans} and J_{all} as cost functions in Case 1, however, is not as successful. This is explained by the fact that actual reductions in the cost functions themselves are only 0·05 and 1·3 dB for J_{trans} and J_{all} respectively (compared to 41·0 and 9·2 dB for Case 2). Thus, from this brief analysis the success of the use of each parameter as the cost function appears to be very much dependent upon the frequency at which the performance is considered. To provide a more practical comparison the average performance over a frequency band is used, as presented in the next section.

The differing levels of success in using each of the four different cost functions is due to the fact that each cost function is a different representation of the same physical vibration. This fact is illustrated in Figure 3, which shows the value of the four parameters without active control over the frequency range 50–350 Hz. All of the parameters show similar responses indicating higher and lower levels of beam vibration, although J_{trans} is seen to be the least consistent.

6. EFFECT ON E_{TOTAL} WHEN MINIMIZING OTHER COST FUNCTIONS

The results from the previous section show that the success of using each cost function to reduce the value of E_{total} is frequency dependent. For Case 2, detailed above, each parameter was minimised (i.e., used as the cost function) and its minimum value for each frequency in the range 50–350 Hz (at 5 Hz intervals) plotted against the same parameter value without active control. Figure 4 shows the results for each of the four cost functions considered.

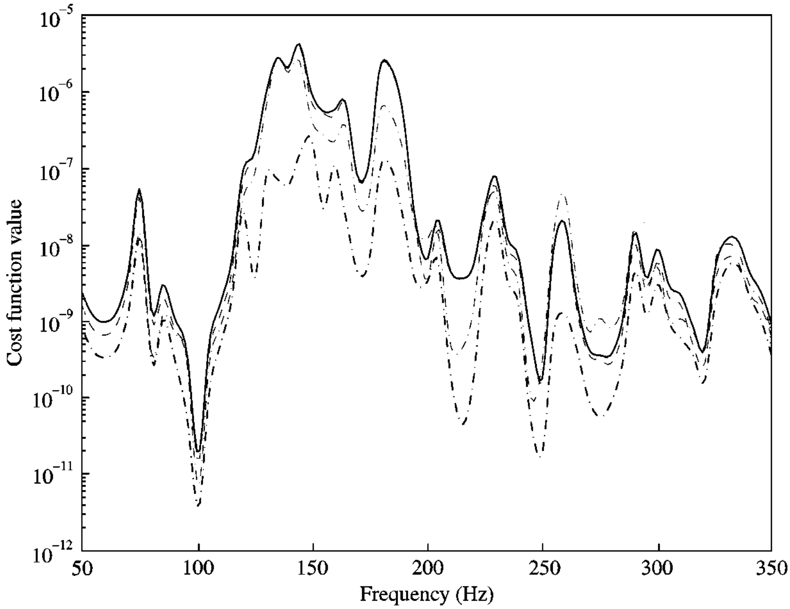

Figure 3. The variation of the four different parameters used to quantify the vibration of beam 40 with excitation frequency when uncontrolled. ——, E_{total}; – – – –, E_{flex}; ·—·—·—·—, J_{trans}; ·–·–·–·–, J_{all}.

Reductions, even though slight in some cases, are achieved at all frequencies within this range for all the parameters. The success of using the other cost functions was evaluated by determining the level of E_{totol} in the beam at each frequency as a consequence of minimizing each cost function. The results are shown in Figure 5, which confirms the frequency dependence suggested in section 5. It is seen that the best reductions in E_{total} are achieved using the two energy-based cost functions. Using J_{trans} as the cost functions actually increases the value of E_{total} (by almost two orders of magnitude) at some frequencies. The average performance of the cost function over the band of frequencies of interest will provide an average measure of the success using each cost function. The frequency-averaged cost functions are defined for a general cost function parameter, CF, as

$$\langle CF \rangle = \frac{1}{n} \sum_{k=1}^{n} CF(\omega_L + (k-1)\Delta\omega), \tag{34}$$

where n is the number of frequency steps, $\Delta\omega$ the angular frequency spacing and ω_L the lower angular frequency point. For all cases subsequently discussed, the circular frequency band range 150–250 Hz is used, comprising of 21 steps, thus $\Delta\omega/2\pi$ is 5 Hz.

6.1. EFFECT OF COST FUNCTION ON OPTIMUM APPLICATIONS OF ACTIVE CONTROL

The success of applying active vibration control is chiefly dependent upon three factors: the position of the actuators, the cost function used and the control system employed. The latter is outside the scope of this paper. The determination of the best actuator positions can be a combinatorially large problem. To guarantee finding the best actuator positions for a given number of actuators, the minimized value of the frequency-averaged cost function

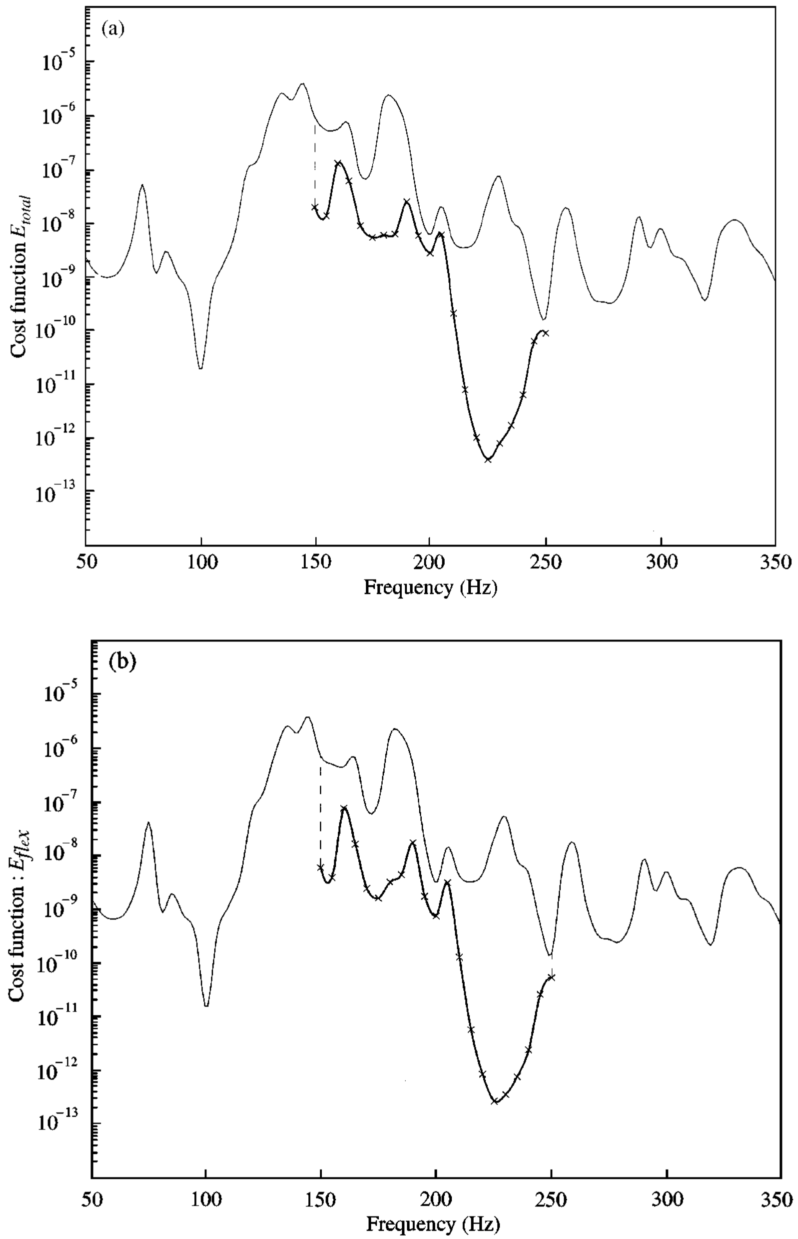

Figure 4. Variation of the four parameters: (a) E_{total}, (b) E_{flex}, (c) J_{trans} and (d) J_{all}, with frequency, and their corresponding minimized values when used as a cost function over the frequency band for which active control is applied, for Case 2.

needs to be determined for all possible combinations of actuator positions. Where this is not feasible other methods are employed, as discussed in the Introduction.

The best actuator position for an AVC system using a particular cost function parameter was achieved by determining the attenuation attainable in the cost function parameter itself for each possible actuator position. This enables a ranking of positions to be achieved and one of the higher ranked combinations of actuator positions is then selected, although practical or other considerations might prevent the best ranked combination from being

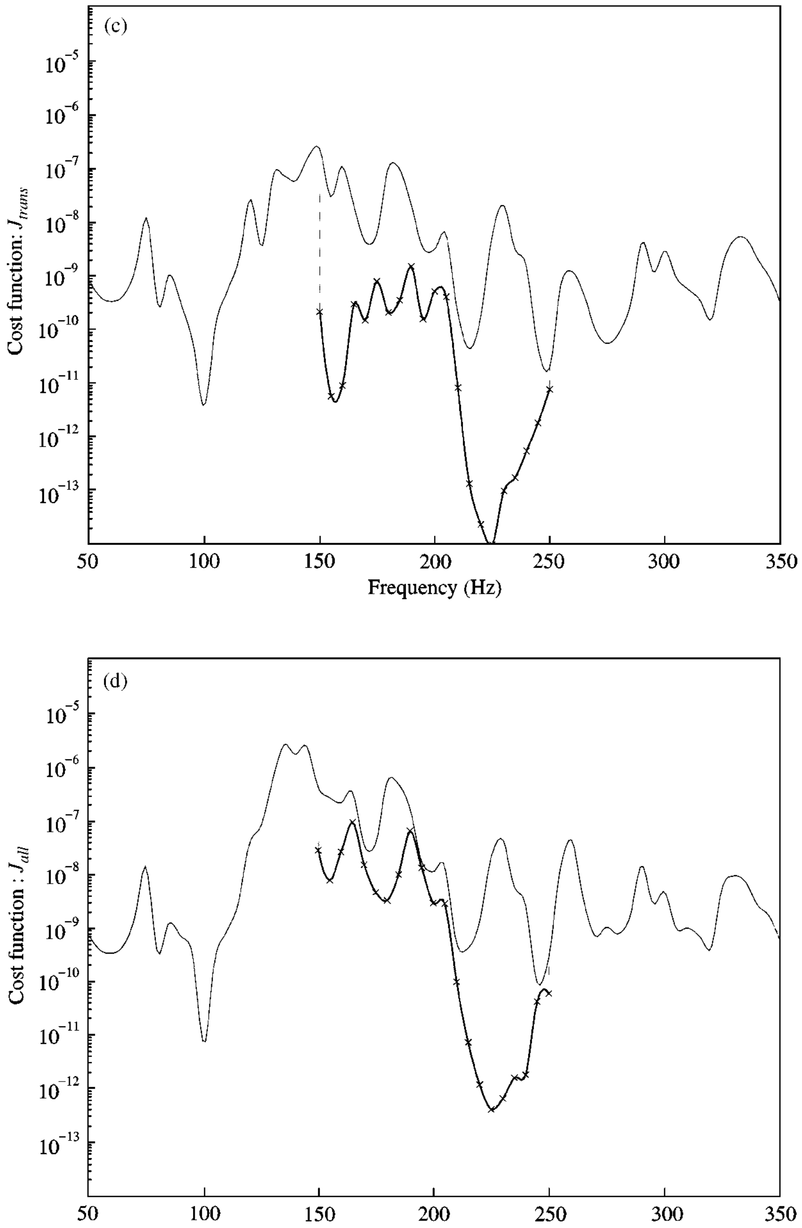

Figure 4. Continued.

utilized. The optimal positions were computed for each of the four cost functions. The best ranked positions were calculated for one, two and three actuators. Beam 40 was not used as a candidate position for an actuator. Hence, there are 39, 741, 9139 possible actuator positions for one, two and three actuators. It is thus feasible to perform an exhaustive search of all possible combinations. Further details on this process are described in reference [7] where the optimal actuator positions were computed for the case where $\langle E_{flex} \rangle$ was used as the cost function.

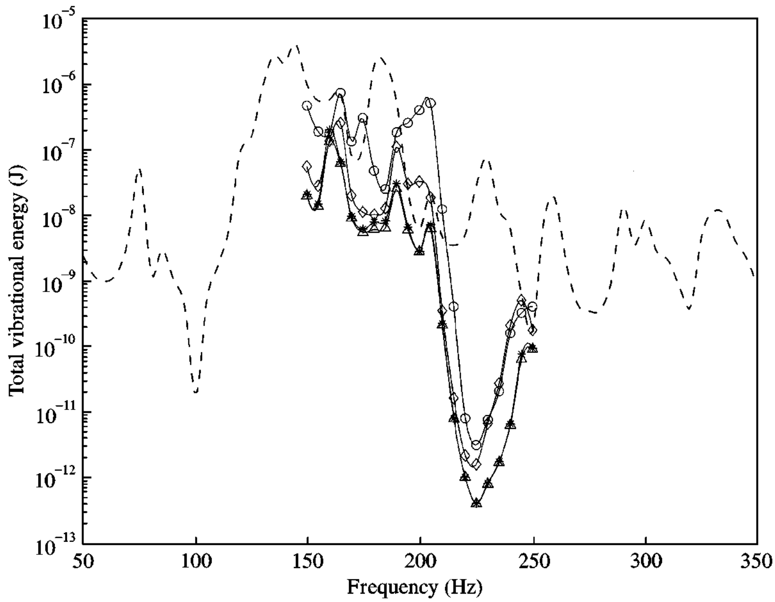

Figure 5. The values of E_{total} produced as a consequence of applying active control with each parameter as cost function, with minimized cost function values as shown in Figure 6. E_{total}: Δ, E_{flex}: $*$, J_{trans}: $\bigcirc$, J_{all}: $\diamond$.

For the best sets of actuator positions, determined using each cost functions parameter, the consequential attenuation of E_{total} was then evaluated. E_{total} has been shown above to be the best representation of the total vibrational energy and hence is used as the reference by which the success of using other cost functions are evaluated. If a particular cost function is a good representation of the total vibrational energy of the beam (E_{total}), then the high and low values of attenuation in the cost function parameter will correspond to high and low values of attenuation in E_{total}. The cost function can then be said to be a *predictable* measure of E_{total}. This will lead to the ranking of actuator positions on the basis of the cost function parameter such that the higher ranked ones will provide the best reduction in E_{total} for the cost function. Also, similar values of attenuation should be achieved when using each cost function as are achieved when using E_{total}.

6.1.1. *Single-actuator active control*

The success of using each of the four frequency-averaged cost functions in an active control system using a single actuator was studied. The results are presented in Figure 6. Each graph shows the consequential attenuation achieved in $\langle E_{total} \rangle$ for each actuator position, which has been ranked in performance of the cost function parameter attenuation. The attenuation for each cost function is shown by the plain line, and is thus monotonically decreasing due to the ranking. It is stressed that each rank number does not necessarily correspond to the same actuator position for each cost function. The best of the cost functions, apart from the reference is $\langle E_{flex} \rangle$ which appears to yield similar reductions to $\langle E_{total} \rangle$. $\langle E_{flex} \rangle$ thus appears to be a predictable measure of $\langle E_{total} \rangle$, so that the actuator positions which give high values of attenuation in $\langle E_{flex} \rangle$ also give high values of attenuation in $\langle E_{total} \rangle$. Next, the use of $\langle J_{all} \rangle$ also provides good attenuation in $\langle E_{total} \rangle$, although this parameter is not such a predictable measure of $\langle E_{total} \rangle$ as $\langle E_{flex} \rangle$. Some of the

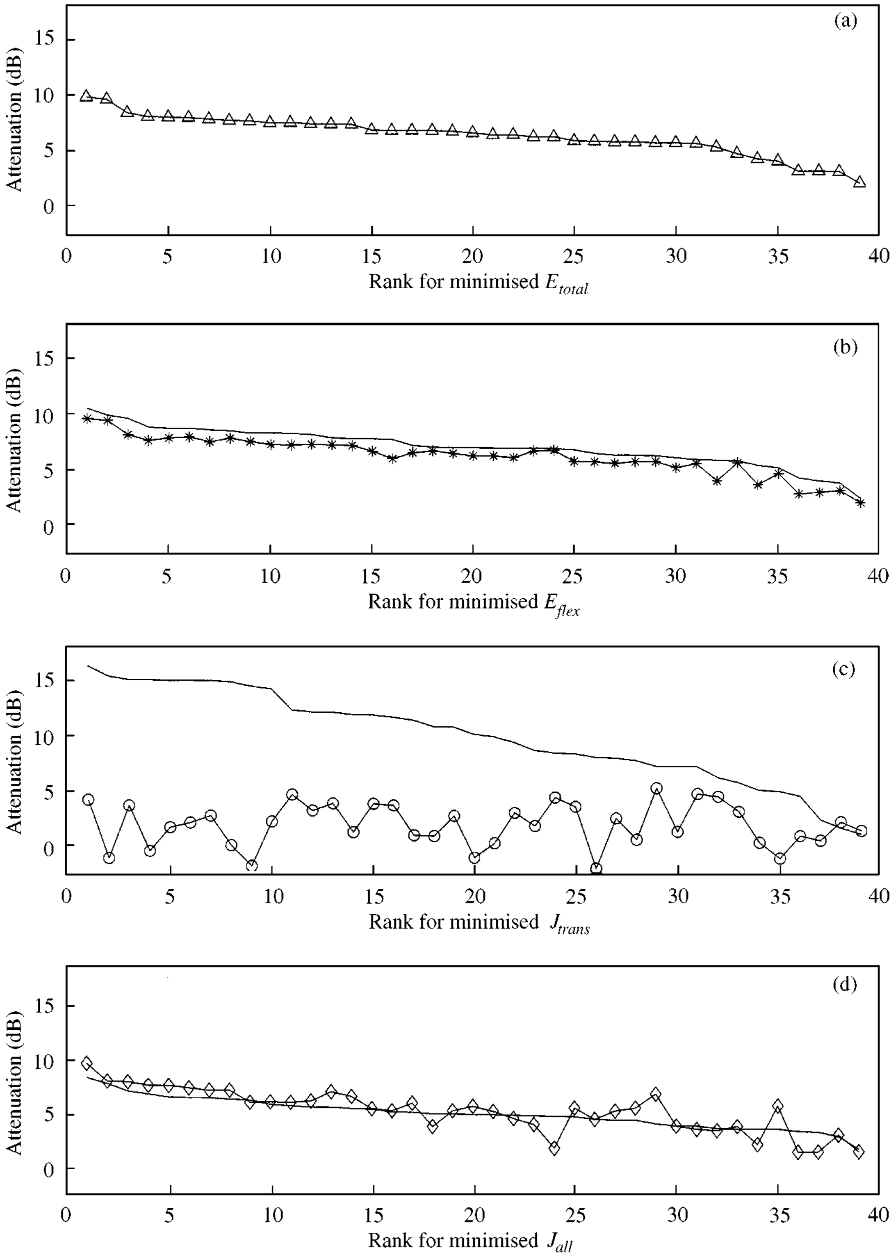

Figure 6. Attenuation achieved in each cost function and E_{total} for all single-actuator positions using each cost function. The actuator positions are ranked in order of decreasing attenuation achieved for each cost function parameter (shown as plain line). The attenuation in E_{total} for each instance is shown when using as the cost function: (a) E_{total}: △, (b) E_{flex}: ∗, (c) J_{trans}: ○ and (d) J_{all}: ◇.

better values of attenuation achieved in $\langle E_{total} \rangle$ are found at low ranked positions and thus would not normally be selected on the basis of the cost function performance. Despite this, the use of this cost function is not disastrous as would be the case with the use of $\langle J_{trans} \rangle$. In this case the ranking obtained on the basis of the cost function is no use for predicting good values of attenuation in $\langle E_{total} \rangle$. Here all the attenuation values in $\langle E_{total} \rangle$ are below 6 dB

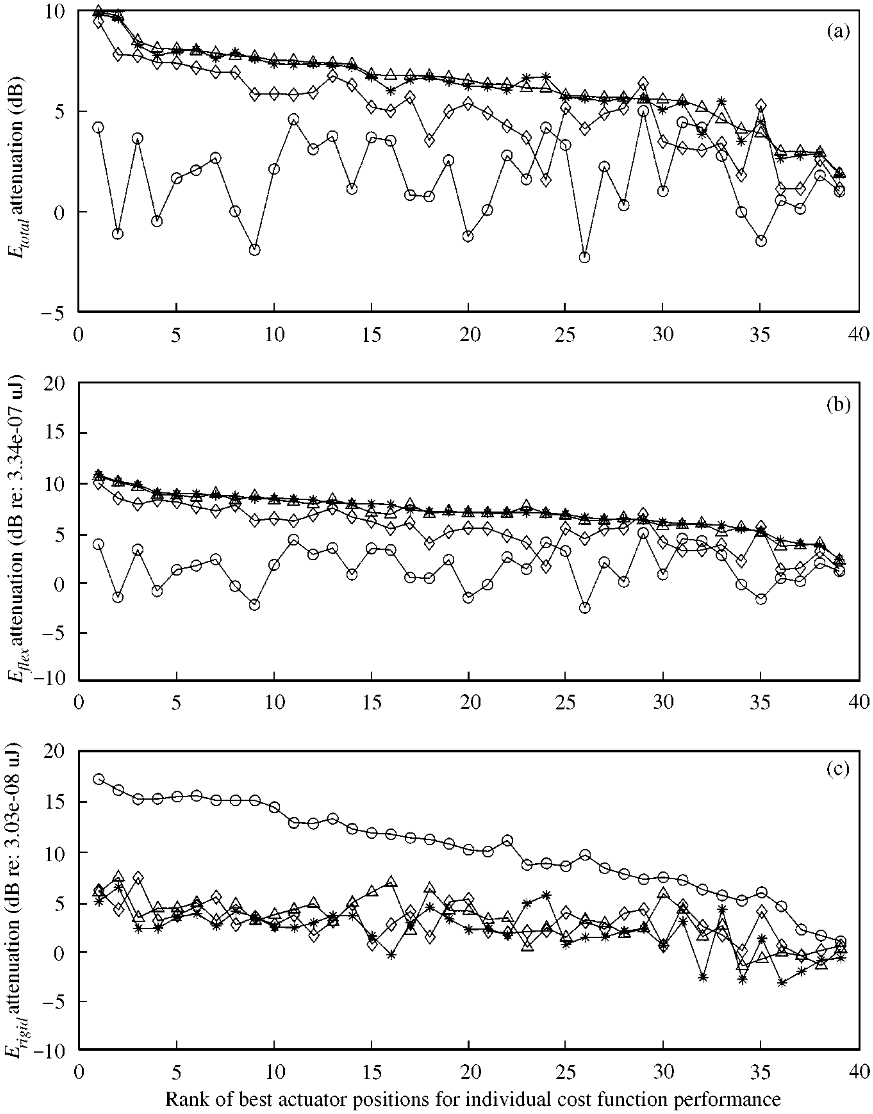

Figure 7. (a) Summary of results in Figure 8: performance achieved in E_{total} by minimizing each cost function parameter with results plotted on common axes and each cost function denoted by same symbols as in Figure 8. Values of attenuation in (b) E_{flex} and (c) E_{rigid} for corresponding combinations in (a) (cost function E_{total}: $\triangle$, E_{flex}: $*$, J_{trans}: $\bigcirc$, J_{all}: $\diamond$).

and in a few cases (including the second ranked position) the use of this cost function actually increases $\langle E_{total} \rangle$. Thus, $\langle J_{trans} \rangle$ is neither a good nor a predictable measure of $\langle E_{total} \rangle$.

To aid comparison between the absolute values of consequential attenuation achieved in $\langle E_{total} \rangle$ for each cost function, all of the values of attenuation in $\langle E_{total} \rangle$ achieved with a single actuator against the individual rankings for each cost function are presented on common axes in Figure 7. It is emphasized that each rank may represent different actuator position combinations for each cost function. To gain a physical insight into why the performance of some cost functions are better than others, the values of consequential attenuation in $\langle E_{total}\rangle$ resulting for each cost function are split into the two constituent parts $\langle E_{flex}\rangle$ and $\langle E_{rigid}\rangle$, as studied in section 5. These are also presented in Figure 7. From the reference values used for the dB scales shown on the axes for each of these components (the energy levels without AVC) it is seen that the significant energy component is $\langle E_{flex}\rangle$. Thus, to achieve good values of attenuation in $\langle E_{total}\rangle$ each cost function needs to produce good values of attenuation in $\langle E_{flex}\rangle$. This is achieved, to differing degrees of success, for all of the cost functions, except $\langle J_{trans}\rangle$, and its poor performance in representing $\langle E_{total}\rangle$ is thus explained. It is interesting to note, however, that the use of $\langle J_{trans}\rangle$ does provide a good and predictable measure of the rigid-body kinetic energy of the beam. The actuator positions which give good reductions in $\langle J_{trans}\rangle$ also provide relatively good reductions in $\langle E_{rigid}\rangle$ (at best 10 dB greater than for minimizing other cost functions) which almost monotonically decrease with the ranking for this cost function.

6.1.2. *Multi-actuator active control*

The investigation was extended to an active control system utilizing two and three actuators. Only the findings using three actuators are presenter here; similar results were found when using two actuators.

The results are presented in Figure 8, in the combined format of Figure 7. The ranking of the x-axis refers, again, to the individual ranking for each of the cost functions, and does not imply common actuator positions at each rank value. It is not feasible to show all the 9139 ranked positions and only the top 100 are shown. The order of success between the cost function parameters in minimizing $\langle E_{total}\rangle$ is similar to that when using a single actuator. $\langle E_{flex}\rangle$ is found to yield very predictable reductions in $\langle E_{total}\rangle$, which are also of similar magnitudes. The second best cost function, again, is $\langle J_{all}\rangle$. In general, it achieves in between

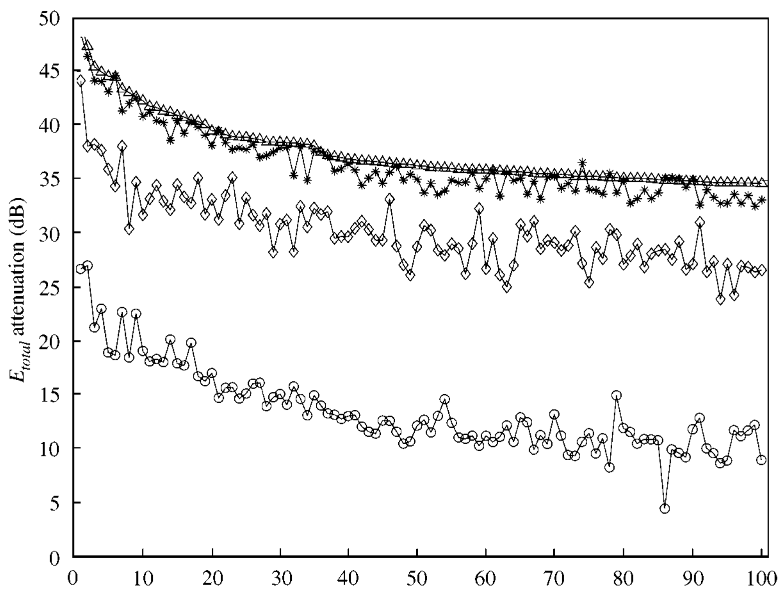

Figure 8. Performance achieved in E_{total} for best ranked three-actuator positions by minimizing each cost function parameter, with results plotted on common axes and each cost function denoted, E_{total}: $\triangle$, E_{flex}: $*$, J_{trans}: $\bigcirc$, J_{all}: $\diamond$.

5 and 10 dB less reduction in $\langle E_{total} \rangle$ than either energy-based cost function. Again, $\langle J_{trans} \rangle$ is the worst in this respect. Although not shown here, it is found that this is due to $\langle J_{trans} \rangle$ not providing good reductions in $\langle E_{flex} \rangle$. However, it still continues to provide large reductions in $\langle E_{rigid} \rangle$, generally 30 dB greater than for other cost functions. The best ranked positions for $\langle J_{trans} \rangle$ yield reductions of over 80 dB in $\langle E_{rigid} \rangle$.

6.2. DISCUSSION OF RESULTS

It is seen for the optimal application of active control over a band of frequencies, which relies on determining the best actuator positions for single and multiple actuators, that there is little difference in using $\langle E_{total} \rangle$ or $\langle E_{flex} \rangle$ as the cost function parameter. E_{total} is, in general, a more comprehensive representation of all types of vibration of the beam. It does not require any parameter measurements which are not already required for E_{flex}. Although the application of active control at single frequencies in section 5 was shown to suggest that E_{total} is the best cost function, especially for cases where either the lack of reduction or increase of E_{rigid} has consequences on the reduction of E_{total}. The frequency-averaged $\langle E_{rigid} \rangle$ is seen to be less significant than $\langle E_{flex} \rangle$, and so generally $\langle E_{flex} \rangle$ is found to perform well as a cost function. It is suggested that this is due to the nature of the beams used in the structure considered here. The beams used are "thin" beams and therefore relatively flexible, and also the natural frequencies for transverse vibration are much lower than those for axial vibration. The first transverse mode occurs at about 240 Hz which is just in the frequency band studied, whereas the first axial mode occurs at about 2·5 kHz. Therefore, the detection of rigid-body motion is thought to be more important for a structure using beams with a greater cross-section (normally termed "rods" or "bars") which only support axial vibration. The development and use of the E_{total} cost function has, however, allowed this to be verified.

Two velocity-based cost functions were also investigated to find their effectiveness in reducing E_{total}. Using only a velocity measurement in the near field of a source has been shown by Pan and Hansen [4] to have worse performance than outside the near field of the source. This is equally applicable to a structural discontinuity, where all the velocity measurements are taken in this case. So the velocity-based cost functions can only be expected to be approximations of E_{total}. It is seen that the incorporation of the rotational velocity components at the ends of each beam is very important to achieve good, predictable reductions in E_{total}, and J_{all} shows a much better performance over simply using the J_{trans} cost function. When using three well-positioned actuators, the J_{all} cost function is seen to have average reductions, over the frequency bandwidth considered, of 5–10 dB less than the E_{total} cost function. For the J_{trans} cost function the attenuation is over 25 dB less. J_{trans}, however, does provide a very good representation of the value of E_{rigid} of the beam, and consistently achieves predictable and much greater reductions than for the other cost functions. Therefore for a structure comprising rigid beams (or rods) the use of only E_{rigid}, which is seen to be well approximated by the slightly simpler J_{trans} parameter, may be sufficient. Thus, the rotational velocity measurement would not be required.

A final note is included on the weighting between the translational and rotational velocity components used in the formulation of J_{all}. The addition of the kinetic energy component due to the rotation of beam 40 was modelled by considering the beam to be composed of two rigid levers whose lengths were half that of the beam, hinged about the beam end. However, this is an approximation, and as only the first flexural modeshape is significant in the frequency range considered, the shape of this mode shape could easily be determined exactly. As the velocity is a function of distance along each half-beam, the net kinetic energy due to flexure of each beam half could therefore be accurately calculated. The approximation used here overestimates the actual kinetic energy of the first mode by a factor of about 3. If the rigid-body kinetic energy component in J_{all} had less significance, a better estimate of E_{total} may well be produced.

7. CONCLUSIONS

The effectiveness of using four different cost functions for an active vibration control system have been studied when applied to a lightweight cantilever 2-D structure comprising thin lightly damped beams. The aim was to reduce the transmission of vibration, in a given frequency band, from the base to the end beam of the structure. Two cost functions were energy-based: the total vibrational energy (E_{total}) and the flexural energy level (E_{flex}). The other two were velocity-based: the sum of the squares of the translational velocity components (J_{trans}) and the weighted sum of the squares of all velocity components (J_{all}). The latter used the rotational velocity component in addition to the translational components at each beam end.

A brief single-frequency analysis showed that the use of E_{flex} as the cost function can result in significant increases in E_{rigid} and so limit of the reduction attainable in E_{total}. E_{total} is confirmed as being the most comprehensive measure of beam vibration and was used as a reference to compare the success of using the other three cost functions. For single-actuator combinations it was found that whilst the frequency-averaged version of E_{total}, $\langle E_{total} \rangle$, is the most comprehensive cost function, it is found that there is little disadvantage in using $\langle E_{flex} \rangle$. This is thought to be because the structure comprises thin "flexible" beams and so bending motion is dominant in the frequency band of interest. Even though the single-frequency cases studied showed some shortcomings of not controlling E_{rigid}, this was not borne out when using cost functions averaged over a frequency band. Generally, reducing $\langle E_{rigid} \rangle$ is thought to be more important if less-flexible beams, or rods, were used as the structural elements.

Using $\langle J_{all} \rangle$ as a cost function was found to be the better velocity-based cost function in reducing $\langle E_{total} \rangle$. For three-actuator AVC systems, the reductions in $\langle E_{total} \rangle$ achieved by minimizing $\langle J_{all} \rangle$ were generally 10 dB less than those achieved by minimizing either $\langle E_{flex} \rangle$ or $\langle E_{total} \rangle$. The use of $\langle J_{trans} \rangle$ as a cost function was not found to yield good reductions in $\langle E_{total} \rangle$. However, it was found to provide a very good prediction of the E_{rigid} component alone, which may prove useful in structures comprising more rigid beams.

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APPENDIX A: MINIMIZATION OF A HERMITIAN QUADRATIC FORM WITH POSITIVE-DEFINITE QUADRATIC COEFFICIENT MATRIX

It is sufficient to show the derivation of the minimum of the cost function used in the main text algebraically, in the case where the quadratic coefficient matrix is positive definite. This assumption avoids the complexities of using differential calculus (as discussed in reference [13], for example) which would normally be required to show a solution for a general case where no such assumptions can be made.

The cost function J is defined in quadratic form with the complex column vector $\mathbf{x}$ containing l complex independent variables,

$$J(\mathbf{x}) = \mathbf{x}^{\mathrm{H}}\mathbf{A}\mathbf{x} + \mathbf{x}^{\mathrm{H}}\mathbf{b} + \mathbf{b}^{\mathrm{H}}\mathbf{x} + c, \tag{A1}$$

where $\mathbf{A}$ is a square matrix of dimension $l \times l$, $\mathbf{b}$ is a complex vector of length l, and c is a positive scalar. If $\mathbf{A}$ is Hermitian and also positive definite, then [14]

$$\mathbf{x}^{\mathrm{H}}\mathbf{A}\mathbf{x} = y > 0 \quad \text{for all } \mathbf{x} \neq \mathbf{0}, \tag{A2}$$

and y will always be a positive scalar, if $\mathbf{x} \neq \mathbf{0}$. The positive definiteness of $\mathbf{A}$ is ensured in practice (see main text) and is verified by testing that all the eigenvalues of $\mathbf{A}$ are positive [14]. Assuming that a solution which minimizes J exists, equation (A1) may be written as

$$J(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0)^{\mathrm{H}}\mathbf{A}(\mathbf{x} - \mathbf{x}_0) + d, \tag{A3}$$

where d is a real scalar, and $\mathbf{x}_0$ is the optimum value of vector $\mathbf{x}$. Expanding equation (A3), so

$$J(\mathbf{x}) = \mathbf{x}^{\mathrm{H}}\mathbf{A}\mathbf{x} - \mathbf{x}^{\mathrm{H}}\mathbf{A}\mathbf{x}_0 - \mathbf{x}_0^{\mathrm{H}}\mathbf{A}\mathbf{x} + c, \tag{A4}$$

allows, firstly, the scalar relation between c and d to be defined as

$$c = \mathbf{x}_0^{\mathrm{H}} \mathbf{A} \mathbf{x}_0 + d. \tag{A5}$$

Secondly, equating the coefficients between equations (A1) and (A4), gives

$$-\mathbf{x}_0 \mathbf{A} = \mathbf{b}, \quad -\mathbf{x}_0^{\mathrm{H}} \mathbf{A} = \mathbf{b}^{\mathrm{H}}, \tag{A6a,b}$$

which are two forms of the same solution. The solution to equation (A3) that minimizes J is clearly given when $\mathbf{x} = \mathbf{x}_0$, and thus

$$J(\mathbf{x}_0) = d = c - \mathbf{x}_0^{\mathrm{H}} \mathbf{A} \mathbf{x}_0. \tag{A7}$$

The optimum values of $\mathbf{x}$ are obtained from equations (A6a,b)

$$\mathbf{x}_0 = -\mathbf{A}^{-1}\mathbf{b}, \quad \mathbf{x}_0^{\mathrm{H}} = -\mathbf{b}^{\mathrm{H}}\mathbf{A}^{-1}. \tag{A8a,b}$$

As $\mathbf{A}$ is positive definite it is also of full rank [14], and hence its inverse exists. As in equation (A6), the two forms given in equation (A8) are not different solutions but equivalent forms of the same solution, since for a Hermitian matrix $\mathbf{A}^{-1} = \mathbf{A}^{-\mathrm{H}}$. The minimum value of the cost function (A7) can be expressed in terms of the coefficients from the quadratic form (A1) using equation (A8),

$$J(\mathbf{x}_0) = c - \mathbf{b}^{\mathrm{H}} \mathbf{A}^{-1} \mathbf{b}. \tag{A9}$$

Because the vector-matrix term in equation (A9) is a positive scalar, the solution is a minimum as the value of $J(\mathbf{x}_0)$ is less than $J(\mathbf{x})$ when $\mathbf{x} = \mathbf{0}$.